1. Noise and Interference
2. Characterization of noise
3. Filtering of noise
4. Filtering of noise - 2
5. Noise bandwidth
6. Quadrature components of noise

Noise and Interference

Introduction to noise and noise filtering.

We have mentioned that communications are, to varying degrees, subject to interference and noise. It's time to be more precise about what these quantities are and how they differ.

Interference represents man-made signals. Telephone lines are subject to power-line interference (in the United States a distorted 60 Hz sinusoid). Cellular telephone channels are subject to adjacent-cell phone conversations using the same signal frequency. The problem with such interference is that it occupies the same frequency band as the desired communication signal, and has a similar structure.

Exercise:

Problem:

Suppose interference occupied a different frequency band; how would the receiver remove it?

Solution:

If the interferer's spectrum does not overlap that of our communications channel—the interferer is out-of-band—we need only use a bandpass filter that selects our transmission band and removes other portions of the spectrum.

We use the notation $i(t)$ to represent interference. Because interference has man-made structure, we can write an explicit expression for it that may contain some unknown aspects (how large it is, for example).

Noise signals have little structure and arise from both human and natural sources. Satellite channels are subject to deep space noise arising from electromagnetic radiation pervasive in the galaxy. Thermal noise plagues **all** electronic circuits that contain resistors. Thus, in receiving small amplitude signals, receiver amplifiers will most certainly add noise as they boost the signal's amplitude. All channels are subject to noise, and we need a way of describing such signals despite the fact we can't write a formula for the noise signal like we can for interference. The most widely used noise model is **white noise**. It is defined entirely by its frequency-domain characteristics.

- White noise has constant power at all frequencies.
- At each frequency, the phase of the noise spectrum is totally uncertain: It can be any value in between 0 and 2π, and its value at any frequency is unrelated to the phase at any other frequency.
- When noise signals arising from two different sources add, the resultant noise signal has a power equal to the sum of the component powers.

Because of the emphasis here on frequency-domain power, we are led to define the **power spectrum**. Because of Parseval's Theorem, we define the power spectrum $P_s(f)$ of a non-noise signal $s(t)$ to be the magnitude-squared of its Fourier transform.
Equation:

$$P_s(f) \equiv (|S(f)|)^2$$

Integrating the power spectrum over any range of frequencies equals the power the signal contains in that band. Because signals **must** have negative frequency components that mirror positive frequency ones, we routinely calculate the power in a spectral band as the integral over positive frequencies multiplied by two.
Equation:

$$\text{Power in}[f_1, f_2] = 2 \int_{f_1}^{f_2} P_s(f)\ \mathrm{d}\ f$$

Using the notation $n(t)$ to represent a noise signal's waveform, we define noise in terms of its power spectrum. For white noise, the power spectrum equals the constant $\frac{N_0}{2}$. With this definition, the power in a frequency band equals $N_0\,(f_2 - f_1)$.

When we pass a signal through a linear, time-invariant system, the output's spectrum equals the product of the system's frequency response and the input's spectrum. Thus, the power spectrum of the system's output is given by

Equation:

$$P_y(f) = (|H(f)|)^2 P_x(f)$$

This result applies to noise signals as well. When we pass white noise through a filter, the output is also a noise signal but with power spectrum $(|H(f)|)^2 \frac{N_0}{2}$.

Characterization of noise

Types of noise:

- Thermal resistor noise: due to thermal agitation. Statistical fluctuations. Always random, erratic and unpredictable.
- Shot noise: fluctuations in emission of electrons or crossing of junctions.
- Additive channel noise: random disturbances added in the channel.
- Multiplicative noise: created by non linear devices like diodes, mixers, power amplifiers.

Terminology:

- Noise is a Random variable best described by statistics like mean, variance, pdf etc.
- Stationary process: always independent of time of measurement - same statistics any time
- Ensemble statistics: measured at the same time on an ensemble (group) of sources
- Ensemble statistics may be different from statistics of any member of group measured over time.
- Ergodic: If ensemble and time statistics are same. ergodic is always stationary but Stationary is not always ergodic.
- Noise from natural phenomenon Gaussian, ergodic & stationary.
- Nonlinear sources may not give Gaussian pdf e.g. rectifier.

Mathematical characterization:

Frequency domain description can be derived for periodic and pulse-like aperiodic phenomena.

- Noise is not repetitive and infinite in time. So not a periodic or aperiodic pulse type signal.
- As an approximation take a sample of noise of interest from -T/2 to T/2 consider it repetitive.
- Purely random so safely assume no DC exists. Now Fourier description applicable.

$$n_T^{(s)}(t) = \sum_{k=1}^{\infty}(a_k \mathrm{cos2\pi k\Delta ft} + b_k \mathrm{sin2\pi k\Delta ft}) = \sum_{k=1}^{\infty} c_k \mathrm{cos(2\pi k\Delta ft} + \theta_k)$$

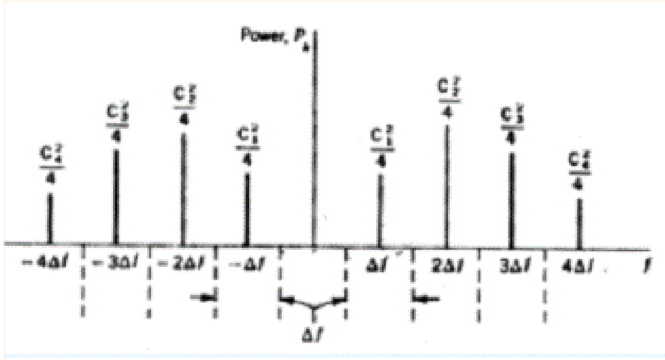
Also,

$c_k^2 = a_k^2 + b_k^2$ and $\theta_k = -\tan^{-1}\frac{b_k}{a_k}$

- Two sided power spectrum and mean power spectral density may be derived

Power associated with each spectral term is
Equation:

$$\frac{c_k^2}{2} = \frac{a_k^2}{2} + \frac{b_k^2}{2}$$



The power spectral density at kΔf is
Equation:

$$G_n(k\mathit{\Delta} f) \equiv G_n(-k\mathit{\Delta} f) \equiv \frac{c_k^2}{4\Delta \mathrm{f}} = \frac{a_k^2 + b_k^2}{4\Delta \mathrm{f}}$$

And total power in Δf at kΔf is
Equation:

$$P_k = 2\mathrm{G}_n(k\mathit{\Delta} f)\mathit{\Delta} f$$

Half of this power is associated with kΔf and other half with -kΔf

- The power spectrum above is deterministic in the sense that it has been derived for a specific waveform, and hence the a,b,c values are specific calculable values. In the general case we can treat these as random variables, replace them by the ensemble average values.

- Let T tend to infinity - and Δf tend to 0. then the actual noise waveform results.

Equation:

$$n(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} (a_k \cos 2\pi k \Delta f t + b_k \sin 2\pi k \Delta f t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} c_k \cos(2\pi k \Delta f t + \theta_k)$$

- Power contribution from coefficients is now replaced be mean square of the random coefficients which vary with chosen T or ensemble member.

Power spectral density can now be defined from mean square coefficients

- The Power spectral density then becomes

Equation:

$$G_n(f) = \lim_{\Delta f \to 0} \frac{c_k^2}{4\Delta f} = \lim_{\Delta f \to 0} \frac{a_k^2 + b_k^2}{4\Delta f}$$

- The power in the frequency range f1 to f2 is given by

Equation:

$$P(f_1 \to f_2) = \int_{-f2}^{-f1} G_n(f) df + \int_{f1}^{f2} G_n(f) df = 2\int_{f1}^{f2} G_n(f) df$$

- And total power PT is

Equation:

$$P_T = \int_{-\infty}^{\infty} G_n(f) df = 2\int_{0}^{\infty} G_n(f) df$$

Filtering of noise

We will see the effects on the power spectrum when noise is passed through a filter.

Filtered output:

If $h(t)$ is the impulse response, filtered output $n_o(t)$ obtained by convolution.
Equation:

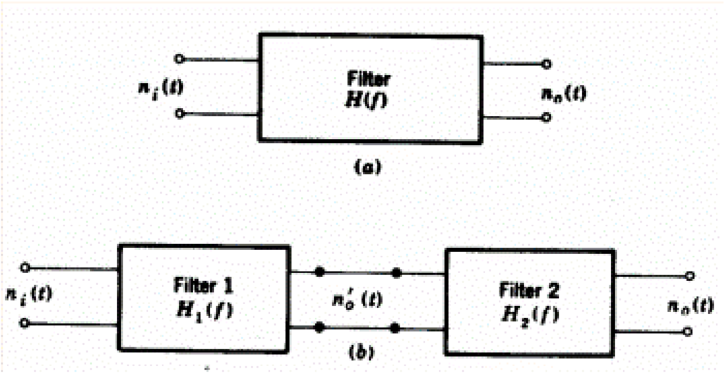
$$n_o(t) = \int_{-\infty}^{\infty} n_i(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} n_i(\tau)h(t-\tau)d\tau$$

- Approximate continuous time by discrete interval.

Equation:

$$n_o(t) = \lim_{\varDelta\tau \to 0} \sum_{k=-\infty}^{k=} n_i(k\varDelta\tau)h(t-k\varDelta\tau)\varDelta\tau$$

- In the interval impulse response is deterministic and known, but n_i depends on sample chosen,
- For white Gaussian noise n_i in interval k is different from and independent of that in interval l.
- Thus output at any time t_0 is a linear superposition of independent Gaussian variables: hence it is Gaussian.
- If we split filter $H(f)$ into $H_1(f)$ and $H_2(f)$, output of $H_1(f)$ is Gaussian but NOT white (due to filtering)



- Hence output n_o of $H_2(f)$ is also non-white.
- output $n_o^{'}$ of $H_1(f)$ is non-white, thus $n_o^{'}$ in intervals k and l are dependent, but despite of this n_o is Gaussian.
- Hence superposition of non-white dependent Gaussian variables also gives Gaussian output.

Spectral components:

- Let spectral component k be

Equation:

$$n_k(t) = a_k \mathrm{cos2\pi k\Delta ft} + b_k \mathrm{sin2\pi k\Delta ft} = c_k \mathrm{cos}(\mathrm{2\pi k\Delta ft} + \theta_k)$$

- Then though coefficients are random over ensemble, they are deterministic for a specific case and depend on the particular sample chosen.
- Hence spectral component is stationary but not ergodic.
- Its normalized power is

Equation:

$$P_k = [n_k(t)]^2 = a_k^2 \mathrm{cos}^2 \mathrm{2\pi k\Delta ft} + b_k^2 \mathrm{sin}^2 \mathrm{2\pi k\Delta ft} + \mathrm{2a}_k b_k \mathrm{cos2\pi k\Delta ftsin2\pi k\Delta ft}$$

- As $n_k(t)$ is stationary, t can be any time. Choose t as $t_1$when cosine is 1, sine is 0. Then

Equation:

$$P_k = a_k^2$$

and similarly

Equation:

$$P_k = b_k^2$$

Equation:

$$P_k = 2\mathrm{G}_n(k\mathit{\Delta} f)\mathit{\Delta} f = -2\mathrm{G}_n(k\mathit{\Delta} f)\mathit{\Delta} f = a_k^2 = b_k^2 = \frac{a_k^2 + b_k^2}{2} = \frac{c_k^2}{2}$$

Since

Equation:

$$a_k^2 = b_k^2$$

- Then p_k can be written as showing a constant term and a sin-cos time dependent term.

Equation:

$$P_k = a_k^2\left(\mathrm{cos}^2 2\pi\mathrm{k}\Delta\mathrm{ft} + \mathrm{sin}^2 2\pi\mathrm{k}\Delta\mathrm{ft}\right) + 2\mathrm{a}_k b_k \mathrm{cos}2\pi\mathrm{k}\Delta\mathrm{ftsin}2\pi\mathrm{k}\Delta\mathrm{ft}$$

thus

Equation:

$$P_k = a_k^2 + 2\mathrm{a}_k b_k \mathrm{cos}2\pi\mathrm{k}\Delta\mathrm{ftsin}2\pi\mathrm{k}\Delta\mathrm{f}$$

- But p_k is stationary and independent of time, so

Equation:

$$a_k b_k = 0$$

Hence the coefficients are seen to be uncorrelated.

- Also at t_1, we have $n_k(t_1) = a_k$. But n_k is Gaussian as it can be considered output of a narrowband filter whose input is Gaussian.

Thus a_k and b_k are Gaussian.

- Also a_k is output of filter in a non zero frequency interval k, it is not DC, thus mean value of $a_k = 0$. Hence coefficients are Zero-mean Gaussian.
- Taking product of two samples $n_k(t)$ and $n_l(t)$ where

Equation:

$$n_k(t) = a_k \mathrm{cos2πkΔft} + b_k \mathrm{sin2πkΔft}$$

Equation:

$$n_l(t) = a_l \mathrm{cos2πlΔft} + b_l \mathrm{sin2πlΔft}$$

The ensemble averages of the result must be independent of time as each component is stationary, hence
Equation:

$$a_k a_l = a_k b_l = b_k a_l = b_k b_l = 0$$

- Hence coefficients at a given frequency are uncorrelated, and also coefficients at different frequencies are uncorrelated.

Summary of noise characteristics:

Random, Gaussian, Ergodic, Linear superposition of random spectral components with coefficients which are Gaussian, zero-mean, with variances related to power spectral density and uncorrelated with each other and with other coefficients at different frequencies.

Filtering of noise - 2

PSD after filtering:

The relation between a ,b and c and φ which describe the noise components can be seen to be identical with that between X,Y and R and θ.

Hence pdf of c is Rayleigh and that of θ is uniform.

$f(c_k) = \frac{c_k}{P_k} e^{-c_k^2/2\mathrm{P}_k} c_k 0$, $f(\theta_k) = \frac{1}{2\pi} - \pi \leq \theta_k \leq \pi$

Let a spectral component of noise be the input to a filter whose transfer function at frequency $k\Delta f$ is
Equation:

$$H(k\Delta f) = \mid H(k\Delta f) \mid e^{j\phi k} = \mid H(k\Delta f) \mid \angle\phi_k$$

The output spectral component of noise is
Equation:

$$n_{\mathrm{ko}}(t) = \mid H(k\Delta f) \mid a_k \cos(2\pi \mathrm{k}\Delta \mathrm{ft} + \phi_k) + \mid H(k\Delta f) \mid b_k \sin(2\pi \mathrm{k}\Delta \mathrm{ft} + \phi_k)$$

The power associated with the input component is
Equation:

$$P_{\mathrm{ki}} = \frac{a_k^2 + b_k^2}{2}$$

As $\mid H(k\Delta f) \mid$is a deterministic function, $[\mid H(k\Delta f) \mid a_k]^2 = \mid H(k\Delta f) \mid^2 a_k^2$

Similarly for b_k, and thus the power associated with noise output is
Equation:

$$P_{\mathrm{ko}} = \mid H(k\Delta f) \mid^2 \frac{a_k^2 + b_k^2}{2}$$

And the power spectral densities are related by

Equation:

$$G_{\mathrm{no}}(f) = \mid H(f) \mid^2 G_{\mathrm{ni}}(f)$$

Where the $k\Delta f$ has been replaced by f as a continuous variable as Δftends to 0.

Superposition of noises:

Noise can be represented as superposition of (orthogonal) harmonics of Δf therefore total power is the result of superposition of component powers.

Consider Two processes n_1 and n_2 with overlapping spectral components.

Power of the sum of n_1 and n_2 will be $p_1 + p_2 + 2E[n_1 n_2]$ and since n_1 and $n_2$are uncorrelated, the last term = 0.

Then these noises also obey the superposition of powers rule.

Mixing of noise with a sinusoid

If k^{th} component of noise is mixed with a sinusoid

Equation:

$$\begin{gathered} n_k(t)\cos 2\pi \mathrm{f}_o t = \frac{a_k}{2}\cos 2\pi(k\Delta f + f_o)t + \frac{b_k}{2}\sin 2\pi(k\Delta f + f_o)t \\ + \frac{a_k}{2}\cos 2\pi(k\Delta f - f_o)t + \frac{b_k}{2}\sin 2\pi(k\Delta f + f_o)t \end{gathered}$$

Sum and difference frequency noise spectral components with 1/2 amplitude are generated and

Equation:

$$G_n(f + f_o) = G_n(f - f_o) = \frac{G_n(f)}{4}$$

Considering power spectral components at $k\Delta f$ and $l\Delta f$, let the mixing frequency be $f_0 = (k + l)\Delta f$. This will generate 2 difference frequency components at the same frequency: $p\Delta f = f_0 - k\Delta f = l\Delta f - f_0$

Then difference frequency components are

Equation:

$$n_{p1}(t) = \frac{a_k}{2}\cos 2\pi p\Delta f t - \frac{b_k}{2}\sin 2\pi p\Delta f t$$

Equation:

$$n_{p2}(t) = \frac{a_l}{2}\cos 2\pi p\Delta f t + \frac{b_l}{2}\sin 2\pi p\Delta f t$$

But as $a_k a_l = a_k b_l = b_k a_l = b_k b_l = 0$, We find $E[n_{p1}(t) n_{p2}(t)] = 0$

and $E\ \ [n_{p1}(t) + n_{p2}(t)]^2\ \ = E\ \ [n_{p1}(t)]^2\ \ + E\ \ [n_{p2}(t)]^2$

Thus superposition of power applies even after shifting due to mixing.

Noise bandwidth

Mixing of noise with a sinusoid

If k^{th} component of noise is mixed with a sinusoid

Equation:

$$n_k(t)\cos 2\pi \mathrm{f}_o t = \tfrac{a_k}{2}\cos 2\pi(k\Delta f + f_o)t + \tfrac{b_k}{2}\sin 2\pi(k\Delta f + f_o)t$$
$$+\tfrac{a_k}{2}\cos 2\pi(k\Delta f - f_o)t + \tfrac{b_k}{2}\sin 2\pi(k\Delta f + f_o)t$$

Sum and difference frequency noise spectral components with 1/2 amplitude are generated and

Equation:

$$G_n(f+f_o) = G_n(f-f_o) = \frac{G_n(f)}{4}$$

Considering power spectral components at $k\Delta f$ and $l\Delta f$, let the mixing frequency be $f_0 = (k+l)\Delta f$. This will generate 2 difference frequency components at the same frequency $p\Delta f = f_0 - k\Delta f = l\Delta f - f_0$

Then difference frequency components are

Equation:

$$n_{p1}(t) = \frac{a_k}{2}\cos 2\pi \mathrm{p\Delta ft} - \frac{b_k}{2}\sin 2\pi \mathrm{p\Delta ft}$$

Equation:

$$n_{p2}(t) = \frac{a_l}{2}\cos 2\pi \mathrm{p\Delta ft} + \frac{b_l}{2}\sin 2\pi \mathrm{p\Delta ft}$$

But as $a_k a_l = a_k b_l = b_k a_l = b_k b_l = 0$

We find

Equation:

$$E[n_{p1}(t)n_{p2}(t)] = 0$$

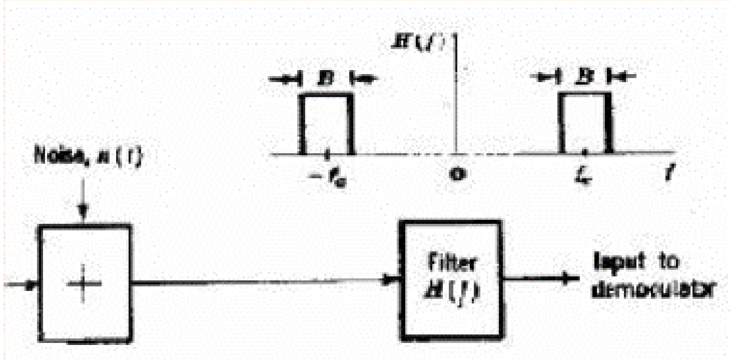
And

Equation:

$$E\left\{[n_{p1}(t) + n_{p2}(t)]^2\right\} = E\left\{[n_{p1}(t)]^2\right\} + E\left\{[n_{p2}(t)]^2\right\}$$

Thus superposition of power applies even after shifting due to mixing.

Minimizing Noise in Systems by filtering:



Assume white noise with $G_n(f) = \frac{\eta}{2}$

To minimize noise entering the demodulator, a filter of bandwidth B can be placed, with B just wide enough to pass signal of interest. Output noise depends on the filter used.

Ideal LPF with white noise has $N_o = \eta B$

rectangular BPF with white noise has $N_o = 2\frac{\eta}{2}(f_2 - f_1) = \eta(f_2 - f_1)$

RC Low Pass Filter:

The filter transfer function is

Equation:

$$H(f) = \frac{1}{1 + j\frac{f}{f_c}}$$

Using $G_{\mathrm{no}}(f) = \mid H(f) \mid^2 G_{\mathrm{ni}}(f)$

we have

Equation:

$$G_{\mathrm{no}}(f) = \frac{\eta}{2}\frac{1}{1 + \left(\frac{f}{f_c}\right)^2}$$

and noise power at filter o/p is

Equation:

$$N_o = \int\limits_{-\infty}^{\infty} G_n(f)\mathrm{df} = \frac{\eta}{2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{df}}{1 + \left(\frac{f}{f_c}\right)^2}$$

using $x = \frac{f}{f_c}$ and noting that $\int\limits_{-\infty}^{\infty}\frac{\mathrm{dx}}{1+x^2} = \pi$, we have

Equation:

$$N_o = \frac{\pi}{2}\eta f_c$$

Differentiating Filter:

The transfer function is

Equation:

$$H(f) = j2\pi\tau f$$

white noise creates output psd

Equation:

$$G_{\mathrm{no}}(f) = 4\pi^2\tau^2 f^2 \frac{\eta}{2}$$

and following this by a rectangular lpf of bandwidth B, noise at o/p is

$$N_o = \int_{-B}^{B} G_{\mathrm{no}}(f)\mathrm{df} = \frac{4\pi^2}{3}\eta\tau^2 B^3$$

Integrating Filter:

An integrator integrating over an interval T has transfer function

Equation:

$$H(f) = \frac{1}{j\omega\tau} - \frac{e^{-j\omega T}}{j\omega\tau}$$

and thus

Equation:

$$\mid H(f) \mid^2 = \left(\frac{T}{\tau}\right)^2 \left(\frac{\sin\pi \mathrm{Tf}}{\pi \mathrm{Tf}}\right)^2$$

The noise power o/p with white noise input

Equation:

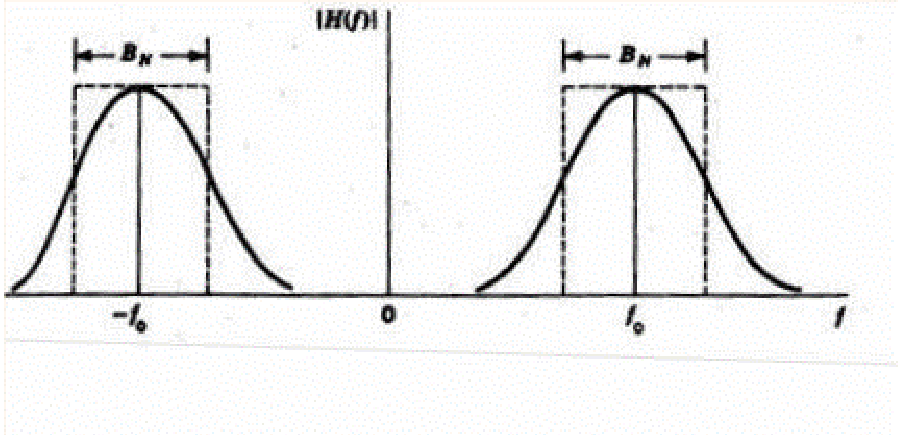
$$N_o = \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \int_{-\infty}^{\infty}\left(\frac{\mathrm{sin}\pi\mathrm{Tf}}{\pi\mathrm{Tf}}\right)^2 \mathrm{df} = \frac{\eta T}{2\tau^2}$$

(The integral has a value = π)

Noise Bandwidth:

If a real filter with transfer fn $H(f)$ centered at f_o is used, we can consider an equivalent rectangular filter centered at f_o with a bandwidth B_N passing the same noise power.

- B_N is called the noise bandwidth of the real filter



- For the RC Filtered curves, the area can be shown to be

Equation:

$$N_o(\mathrm{RC}) = \frac{\pi}{2}\eta f_c$$

- For a rectangular filter we have

Equation:

$$N_o(\text{rect}) = \frac{\eta}{2}2\text{B}_N = \eta B_N$$

- Setting $N_0(\text{RC}) = N_0(\text{rect})$, $B_N = \frac{\pi}{2} f_c$

Hence noise BW of RC filter is 1.57 times its 3 dB BW.

Quadrature components of noise

Narrowband Representation

When passed through a narrowband filter noise can also be represented in terms of quadrature components

Equation:

$$n(t) = n_c(t)\cos 2\pi f_o t - n_s(t)\sin 2\pi f_o t$$

Where f_o corresponds to $k = K$ and lies in the centre of the band.

Letting $f_o = K\Delta f$and using $2\pi f_o t - 2\pi K\Delta f t = 0$

We add the above to the arguments in the equation 1.

Equation:

$$n(t) = \lim_{\Delta f\to 0}\sum_{k=1}^{\infty}\{a_k\cos 2\pi[f_0 + (k-K)\Delta f]t + b_k\sin 2\pi[f_0 + (k-K)\Delta f]t\}$$

Using identities for the sine and cosine of sum of two angles, we get the equation in terms of n_c and n_s, where

Equation:

$$n_c(t) = \lim_{\Delta f\to 0}\sum_{k=1}^{\infty}[a_k\cos 2\pi(k-K)\Delta f t + b_k\sin 2\pi(k-K)\Delta f t]$$

Equation:

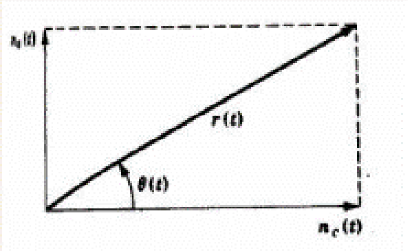
$$n_s(t) = \lim_{\Delta f\to 0}\sum_{k=1}^{\infty}[a_k\sin 2\pi(k-K)\Delta f t - b_k\cos 2\pi(k-K)\Delta f t]$$

n_c and n_s are stationary random processes represented as superposition of components.

It can be shown that n_c and $n_s$are Gaussian, zero mean equal variance uncorrelated variables.

Significance:

$n(t)$of frequency $k\Delta f$gives rise to n_c and n_s of frequency $f - f_o$within band -B/2 to B/2 and thus change insignificantly during one $f_o$cycle. n_c and n_s represent amplitude variations of two slowly changing quadrature phasor components, the complete phasor is



$$r(t) = \left[n_c^2(t) + n_s^2(t)\right]^{\frac{1}{2}}$$

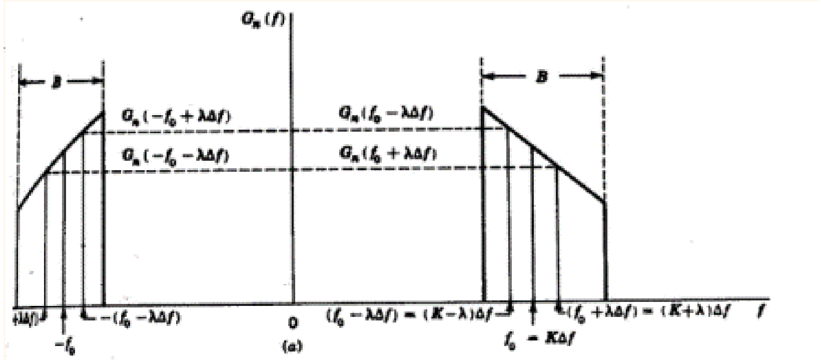
Equation:

$$\theta(t) = \tan^{-1}[n_s(t)/n_c(t)]$$

The endpoint of r wanders randomly with passage of time.

PSD of orthogonal noise

Select spectral components corresponding to $k = K + \lambda$ and $k = K - \lambda$ where λ is an integer. $k = K$corresponds to frequency f_o, hence selected components correspond to $f_o + \lambda\Delta f$ and $f_o - \lambda\Delta f$, and these frequencies generate 4 power spectral lines.


$G_n(f)$
B
B
$G_n(-f_o + \lambda\Delta f)$
$G_n(-f_o - \lambda\Delta f)$
$G_n(f_o - \lambda\Delta f)$
$G_n(f_o + \lambda\Delta f)$
$+\lambda\Delta f)$
$-f_o$
$-(f_o - \lambda\Delta f)$
0
(a)
$(f_o - \lambda\Delta f) = (K-\lambda)\Delta f$
$f_o = K\Delta f$
$(f_o + \lambda\Delta f) = (K+\lambda)\Delta f$
f

Select from $n_c(t)$ that part $\Delta n_c(t)$corresponding to the frequencies we have selected above

$\mathit{\Delta}n_c(t) = a_{K-\lambda}\text{cos2πλ}\mathit{\Delta}\text{ft} - b_{K-\lambda}\text{sin2πλ}\mathit{\Delta}\text{ft} + a_{K+\lambda}\text{cos2πλ}\mathit{\Delta}\text{ft} + b_{K+\lambda}\text{sin2πλ}\mathit{\Delta}\text{ft}$

All 4 terms are at same frequency and represent uncorrelated processes. Then the power P_λ of $\mathit{\Delta}n_c(t)$ is the ensemble average of $[\mathit{\Delta}n_c(t)]^2$ and this may be calculated at any time t_1. Choosing t_1 such that $\lambda\mathit{\Delta}\text{ft}_1$ is an integer,

Equation:

$$\mathit{\Delta}n_c(t) = a_{K-\lambda} + a_{K+\lambda}$$

Equation:

$$P_\lambda = E\left\{[\mathit{\Delta}n_c(t_1)]^2\right\} = E\left[(a_{K-\lambda} + a_{K+\lambda})^2\right] = a^2_{K-\lambda} + a^2_{K+\lambda}$$

We then find

Equation:

$$P_\lambda = 2\text{G}_{\text{nc}}(\lambda\mathit{\Delta}f)\mathit{\Delta}f = 2\text{G}_n[(K-\lambda)\mathit{\Delta}f]\mathit{\Delta}f = 2\text{G}_n[(K+\lambda)\mathit{\Delta}f]\mathit{\Delta}f$$

So that

Equation:

$$G_{\text{nc}}(\lambda\mathit{\Delta}f) = G_n[(K-\lambda)\mathit{\Delta}f] = G_n[(K+\lambda)\mathit{\Delta}f]$$

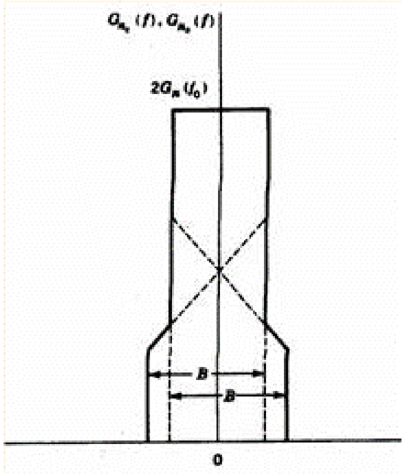
For continuous frequency variable, this becomes

Equation:

$$G_{\text{nc}}(f) = G_n(f_0 - f) + G_n(f_0 + f)$$

Similar equation also results for G_{ns}The psd's of the orthogonal components are shown below, and are obtained by shifting the +ve and -ve parts of plot of $G_n$from $+f_o$and $-f_o$to $x = 0$ and adding the displaced plots.

Thus the power of $n(t)$and the orthogonal components are equal



Example: For white noise filtered by a rectangular BPF centered at f_o with $\mathrm{BW} = B$,
Equation:

$$G_n(f) = \tfrac{\eta}{2} f_o - \tfrac{B}{2} \leq \mid f \mid \leq f_o + \tfrac{B}{2},$$
$$G_n(f) = 0\mathrm{elsewhere}$$

Hence $G_n(f_o + f) = G_n(f_o - f)$ and
Equation:

$$G_{\mathrm{nc}}(f) = G_{\mathrm{ns}}(f) = G_n(f_o - f) + G_n(f_o + f) = \frac{\eta}{2} + \frac{\eta}{2} = \eta \mid f \mid \leq \frac{B}{2}$$

and are twice the magnitude of $G_n(f_o + f)$.

The power of the orthogonal components is:
Equation:

$$\sigma_{\mathrm{nc}}^2 = \sigma_{\mathrm{ns}}^2 = \int_{-B/2}^{B/2} G_{\mathrm{nc}}(f) \mathrm{df} = \eta B$$

And that of $n(t)$ is
Equation:

$$\sigma_n^2 = \int_{-\mathrm{fo}-B/2}^{-\mathrm{fo}+B/2} G_n(f)\mathrm{df} + \int_{\mathrm{fo}-B/2}^{\mathrm{fo}+B/2} G_n(f)\mathrm{df} = 2\frac{\eta}{2}B = \eta B$$